Physics 231: General Relativity
Final Exam: May 13-23, 2003
Ori Ganor

Directions

- Please solve both problems below.
- Please explain your solution.
- Even if you do not reach a final answer, please explain your plan for the solution and please write down the relevant formulas. That may help for partial credit.
- No need to rederive standard expressions that we derived in the classroom or that appear in the textbook.
- **Please return your solutions to me no later than 48 hours after you have picked the exam, and in any case no later than 5pm on Friday, May 23, 2003.**
- When you are finished, you can either return a handwritten solution to Birge Hall 445 (slide it under the door if there is no one there), or email the solutions to

  `origa@socrates.berkeley.edu`

  I can read *TeX, LaTeX, MSWord* and *PDF*. (Please don't send Mathematica notebook files.) If you choose to return a handwritten solution, it would be good if you could also email me a message so as to be sure that I got your solution.
- During the exam period, you can communicate with me via e-mail.
- The maximal number of points that you can score for each problem is indicated in brackets $[\cdots]$. These numbers are *tentative* and might change slightly.

Good luck!

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Problems

1 Problem 1: Black Holes [50pt]

(A) [15pt] A stationary observer sits at $r = r_0 > 2M$ in the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

The observer looks at the sky and finds that the light rays from the universe outside the black hole come only through a stereographic angle of A (out of the full 4π). Find A in terms of r_0.

(B) [15pt] Now we study a similar question for an observer that crosses the event horizon. Assume that the observer in part (A) disengages the engine that holds it in place and starts freely falling towards the blackhole along a radial direction. We use the Kruskal coordinates

$$v = \left(\tfrac{r}{2M} - 1\right)^{\frac{1}{2}} e^{\frac{r}{4M}} \sinh\tfrac{t}{4M}, \quad u = \left(\tfrac{r}{2M} - 1\right)^{\frac{1}{2}} e^{\frac{r}{4M}} \cosh\tfrac{t}{4M}, \quad \text{for } r > 2M$$
$$v = \left(1 - \tfrac{r}{2M}\right)^{\frac{1}{2}} e^{\frac{r}{4M}} \cosh\tfrac{t}{4M}, \quad u = \left(1 - \tfrac{r}{2M}\right)^{\frac{1}{2}} e^{\frac{r}{4M}} \sinh\tfrac{t}{4M}, \quad \text{for } r < 2M$$

The metric is

$$ds^2 = \frac{32M^3}{r} e^{-\frac{r}{2M}}(du^2 - dv^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Assume that the horizon is crossed at the point $u = v = u_0 > 0$ and that at that point $\frac{du}{d\tau} = 0$, where τ is the proper time of the infalling observer. At this point the observer looks up to the sky. What is the stereographic angle that is formed by light from the outside universe now?

(C) [10pt] What is the relation between u_0 and r_0 of parts (B) and (A)?

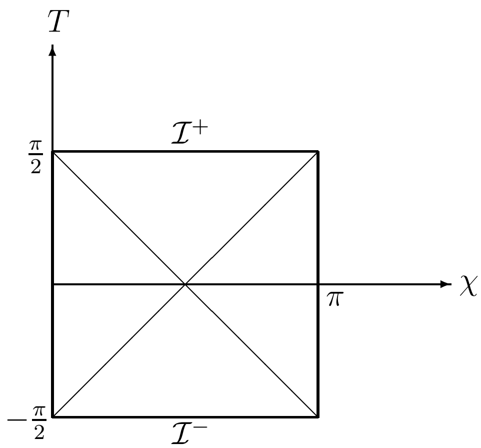
(D) [10pt] The observer at $u = v = u_0$ (and $\frac{du}{d\tau} = 0$ as above) now looks straight up in the radial direction away from the black hole and sees a light ray with frequency ν_o. Assuming that the light ray originated from a stationary emitter at $r = R \gg 2M$ far outside the blackhole, what was the frequency ν_e of the light ray when it was emitted? (Don't forget that the observer is moving in the (r, t) coordinate system!)

2 Problem 2: Cosmology [50pt]

De-Sitter space is given by the following metric

$$ds^2 = -d\tilde{t}^2 + \frac{3}{\Lambda}\cosh^2\sqrt{\frac{\Lambda}{3}}\,\tilde{t}\,(d\chi^2 + \sin^2\chi\, d\Omega^2), \qquad d\Omega^2 \equiv d\theta^2 + \sin^2\theta\, d\phi^2.$$

The corresponding Penrose diagram is:



where

$$T = 2\arctan e^{\sqrt{\frac{\Lambda}{3}}\,\tilde{t}} - \frac{\pi}{2} \Longrightarrow \frac{1}{\cos T} = \cosh\sqrt{\frac{\Lambda}{3}}\,\tilde{t}, \qquad -\frac{\pi}{2} < T < \frac{\pi}{2}.$$

As we have seen in class, there are several different parameterizations for de-Sitter space. In this problem we will study another one. It is called the "static" coordinate system.

$$ds^2 = -\left(1 - \frac{\Lambda}{3}r^2\right)dt^2 + \left(1 - \frac{\Lambda}{3}r^2\right)^{-1} dr^2 + r^2 d\Omega^2, \qquad 0 \le r < \sqrt{\frac{3}{\Lambda}}, \qquad -\infty < t < \infty.$$

(A) [25pt] Find the coordinate transformation

$$\tilde{t} \equiv \tilde{t}(t, r), \qquad \chi \equiv \chi(t, r)$$

that relates the two coordinate systems. (Implicit expressions will suffice. You may also set $\Lambda = 3$ if it is more convenient.)

(B) [10pt] The static coordinate system only covers a part of the full de-Sitter space. Draw this part on the Penrose diagram.

(C) [15pt] For $r \ll \Lambda^{-\frac{1}{2}}$, compare the static metric above to the Newtonian approximation

$$ds^2 = -(1 + 2\phi(\tilde{r}))dt^2 + (1 - 2\phi(\tilde{r}))(d\tilde{r}^2 + \tilde{r}^2 d\Omega^2)$$

where $\phi(\tilde{r}) = -\frac{\Lambda}{6}\tilde{r}^2$ is the Newtonian potential corresponding to a uniform mass density. Neglect all terms of order Λ^2 and higher and allow for a change of variables

$$r = \tilde{r} + C\Lambda\tilde{r}^3 + \cdots$$

with some constant C. Show that there is **no** choice of C for which the metrics completely agree (to first order in Λ)! Can you explain why this is so? In other words, what is the difference between a small cosmological constant and a small uniform mass density for which the approximate expression for the metric $ds^2 = -(1+2\phi)dt^2 + (1-2\phi)(d\tilde{r}^2 + \tilde{r}^2 d\Omega^2)$ should hold?

You might need the identities

$$\sin(x+y) + \sin(x-y) = 2\sin x\cos y, \qquad \sin(x+y) - \sin(x-y) = 2\cos x\sin y,$$

$$\frac{\tan(x+y)}{\tan(x-y)} = \frac{\sin 2x + \sin 2y}{\sin 2x - \sin 2y}, \qquad \tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x\,\tan y}$$

and

$$\cosh x = \frac{1}{\sin 2\arctan e^x}, \qquad \sin 2\arctan e^x = \frac{1}{\cosh x}.$$